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On higher-order Riccati equations as Bäcklund transformations

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Abstract. In this short paper we illustrate by few examples the special role played by higher-order Riccati equations in the construction of Bäcklund transformations for integrable systems.

1. Introduction

The Riccati equation [1]

$$\frac{\mathrm{d}v}{\mathrm{d}x}(x) = \alpha(x) + \beta(x)v(x) + v^2(x) \tag{1}$$

the simplest nonlinear ordinary differential equation, plays a very important role in the solution of integrable nonlinear partial differential equations. These equations are characterized by being the compatibility conditions between two linear partial differential equations (the Lax pair) for an auxiliary function, the so-called wavefunction [2]. Among the consequences of the existence of a Lax pair is the fact that one can obtain for them a denumerable number of exact solutions, the so-called soliton solutions. The soliton solutions and their superpositions can be obtained recursively as solutions of the appropriate Bäcklund transformation, a differential relation between two different solutions of the nonlinear equation, starting from a trivial, in general constant, solution of the nonlinear partial differential equation. The best known integrable nonlinear partial differential equation is the Korteweg–de Vries equation (KdV)

$$u_t = u_{xxx} + 6uu_x \qquad v_x = u \tag{2}$$

whose simplest Bäcklund transformation is given by (1).

The Riccati equation, though a nonlinear equation, is characterized by the fact of possessing a superposition formula, as it is the case for all linear equations and using the Cole–Hopf transformation [3] we can reduce it to a linear Schrödinger equation. By using the Cole–Hopf transformation one can obtain a whole class of nonlinear ordinary differential equations, which possesses the same kind of properties as the Riccati equation, the so-called Riccati chain [4].

In this paper we will show that also the higher-order members of the Riccati chain play the role of Bäcklund transformations for nonlinear integrable partial differential equations.

In section 2 we will review the known results on the Riccati equation and its chain, while section 3 is devoted to examples of equations of the Riccati chain, which appear as Bäcklund transformations for the Sawada–Kotera, the Tzitzeica and the Fritzhugh–Nagumo equations. In section 4 a few concluding remarks and comments are presented.

2. The Riccati equation and its chain

The Riccati equation (1) is the simplest nonlinear differential equation which, being linearizable, can be completely solved. It is the only first-order nonlinear ordinary differential equation which possesses the Painlevé property [1], i.e. which has no movable singularity. Moreover, as was shown by Lie and Scheffers [13], the Riccati equation is the only ordinary nonlinear differential equation of first order which possesses a (nonlinear) superposition formula

$$\hat{v}(x) = \frac{v_1(x)(v_3(x) - v_2(x)) + kv_2(x)(v_1(x) - v_3(x))}{v_3(x) - v_2(x) + k(v_1(x) - v_3(x))} \tag{3}$$

i.e. the solution $\hat{v}(x)$ is nonlinearly expressed in terms of $v_1(x)$, $v_2(x)$, $v_3(x)$ which are given solutions of the same Riccati equation (1) and k is a constant parameter. The existence of a nonlinear superposition formula allows one to construct a denumerable set of solutions starting from three given solutions.

Most of the properties of the Riccati equation are also shared by the Riccati chain. The N-order equation of the Riccati chain is given by the following formula:

$$L^N v(x) + \sum_{j=1}^{N} \alpha_j(x)\left(L^{j-1}v(x)\right) + \alpha_0(x) = 0 \tag{4}$$

where N is an integer characterizing the order of the Riccati equation in the chain, L is the following differential operator:

$$L = \frac{\mathrm{d}}{\mathrm{d}x} + cv(x) \tag{5}$$

and $\alpha_j(x)$, $j = 0, 1, \ldots, N$ are $N + 1$ arbitrary functions. The lowest-order equations in the chain after Riccati equation (1) are

$$N = 2: \qquad \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + \left[\alpha_2(x) + 3cv(x)\right]\frac{\mathrm{d}v}{\mathrm{d}x} + c^2v^3(x) + c\alpha_2(x)v^2(x) + \alpha_1(x)v(x) + \frac{\alpha_0(x)}{c} = 0 \tag{6}$$

$$\begin{aligned} N = 3: \qquad & \frac{\mathrm{d}^3 v}{\mathrm{d}x^3} + \left[\alpha_3(x) + 4cv(x)\right]\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + 3c\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 \\ & +\left[6c^2v^2(x) + 3cv(x)\alpha_3(x) + \alpha_2(x)\right]\frac{\mathrm{d}v}{\mathrm{d}x} \\ & +c^3v^4(x) + c^2\alpha_3(x)v^3(x) + c\alpha_2(x)v^2(x) + \alpha_1(x)v(x) + \alpha_0(x) = 0. \end{aligned} \tag{7}$$

Let us notice that the N-Riccati chain is a polynomial expression in $v(x)$ and its derivatives are such that the coefficient of the $(N-1)$-derivative of v is linear in v, that of the $(N-2)$-derivative is quadratic in v and that of zeroth-order derivative is a polynomial in v of order $N + 1$.

By the Cole–Hopf transformation

$$cv(x)\,\psi(x) = \frac{\mathrm{d}\psi}{\mathrm{d}x}(x) \tag{8}$$

the whole class of equations (4) linearizes to a linear ordinary differential equation with variable coefficients of order $N + 1$,

$$\sum_{j=0}^{N} \alpha_j(x) \frac{\mathrm{d}^j \psi}{\mathrm{d}x^j} + \frac{\mathrm{d}^{N+1}\psi}{\mathrm{d}x^{N+1}} = 0. \tag{9}$$

Observe that by a trivial transformation the linear equations (9) can be always put in the Gel'fand–Dickey form [7].

One can show, using (8) and (9) that a nonlinear superposition formula can be derived for any equation of the Riccati chain. Moreover, the whole Riccati chain possesses the Painlevé property; for example, the second-order Riccati equation is equivalent to equation VI in the Ince classification of the equations possessing the Painlevé property (p 334 of [1]).

When the N-Riccati equation represents a Bäcklund transformation then the coefficients α_j also depend on the 'time' variable t in a parametric way and are expressed in terms of a known solution of the given nonlinear partial differential equation. When the given solution is constant then the linear equation (9) has constant coefficients and generically its solution $\psi(x)$ is written out as a combination of exponential functions. These exponential functions are the main ingredients in the construction of the soliton solutions of the N-Riccati equation (4).

3. Examples of applications

In this section we consider few examples of nonlinear evolution equations, which have as Bäcklund transformations a higher Riccati equation. As one can deduce from equations (4) and (9), the nonlinear equations are obtained from the compatibility of a Lax pair given by linear operators of order greater than two.

3.1. The Sawada–Kotera equation

The Sawada–Kotera equation [8] is the nonlinear partial evolution equation

$$u_t = u_{5x} + 10(uu_{xxx} + u_x u_{xx}) + 20u^2 u_x \qquad u = u(x,t). \tag{10}$$

This equation has the same terms as the higher-order KdV equation [2],

$$u_t = u_{5x} + 10(uu_{xxx} + 2u_x u_{xx}) + 30u^2 u_x \qquad u = u(x,t) \tag{11}$$

but has different constant coefficients. Rewriting equations (10) and (11) in terms of a potential

$$v(x,t) = \int_x^\infty u(y,t)\,\mathrm{d}y$$

we can easily verify that the highest-order KdV equation (11) has the lowest-order Bäcklund transformation expressed by a 1-Riccati equation, while for the Sawada–Kotera equation such a Bäcklund transformation is given by a 2-Riccati equation (6) with

$$\begin{aligned} &c = -\tfrac{1}{3} \qquad \alpha_2 = \hat{v}(x,t) \qquad \alpha_1 = \tfrac{1}{3}\hat{v}^2 - \hat{v}_x \\ &\alpha_0 = \hat{v}\hat{v}_x - \hat{v}_{xx} - \tfrac{1}{9}\hat{v}^3 - \mu \end{aligned} \tag{12}$$

where $\hat{v}$ is any solution of the potential Sawada–Kotera equation and μ is an arbitrary constant, the Bäcklund parameter.

The above result corresponds to the fact that the 'space part' of the Lax pair for equation (10) is given by a third-order spectral problem. Such a spectral problem can be easily obtained from the Bäcklund transformation using the following Cole–Hopf transformation:

$$\hat{v} = v - 3\frac{\psi_x}{\psi} \tag{13}$$

and reads

$$\psi_{xxx} + \hat{v}\psi_{xx} + \left(\tfrac{1}{3}\hat{v}^2 - \hat{v}_x\right)\psi_x + \left(\hat{v}\hat{v}_x - \hat{v}_{xx} - \tfrac{1}{9}\hat{v}^3 - \mu\right)\psi = 0. \tag{14}$$

For constant $\hat{v}$, a real solution of equation (14) is given by

$$\begin{aligned}\psi(x,t) = {} & A(t)\exp\left[\mu + \tfrac{1}{9}\hat{v}^3\right]^{3/2}x \\ & + B(t)\exp\left[-\left(\mu + \tfrac{1}{9}\hat{v}^3\right)^{3/2}\right]x\cos\left[\tfrac{1}{2}((\mu + \hat{v}^3/9)\sqrt{3})x + \varphi(t)\right].\end{aligned} \tag{15}$$

From (13) and (15) one obtains a soliton solution of the potential Sawada–Kotera equation,

$$\begin{aligned}v = {} & -3\alpha^2(x)\left\{A(t)\,\mathrm{e}^{\alpha(x)} + B(t)\,\mathrm{e}^{-\alpha(x)}\cos\left[\tfrac{1}{2}\sqrt{3}\,\alpha(x) + \varphi(t)\right]\right\}^{-1}\left\{A(t)\,\mathrm{e}^{\alpha(x)}\right. \\ & \left. - B(t)\,\mathrm{e}^{-\alpha(x)}\left[\cos\left(\tfrac{1}{2}\sqrt{3}\alpha(x) + \varphi(t)\right) + \tfrac{1}{2}\sqrt{3}\sin\left(\tfrac{1}{2}\sqrt{3}\alpha(x) + \varphi(t)\right)\right]\right\}\end{aligned} \tag{16}$$

where $\alpha(x) := (\mu + 1/9\tilde{v}^3)^{3/2}x$. The solution (16) is a solution of the 2-Riccati equation. The requirement that solution (16) satisfies the potential Sawada–Kotera equation determines A and B as functions of the 'time' variable.

3.2. *The Tzitzeica equation*

Consider the case of the Tzitzeica–Bullough–Dodd–Zhiber–Shabat [9] equation in the rational form

$$vv_{xt} - v_x v_t - v^3 + 1 = 0. \tag{17}$$

The equation (17) is consistent with 2-Riccati equation (6) iff

$$\alpha_2 = 0 \qquad c = 1 \qquad \alpha_1 = \frac{\hat{v}_{xx}}{\hat{v}} \qquad \alpha_0 = \lambda \in \mathbb{R}$$

where $\hat{v}$ is another solution of the Tzitzeica equation in the rational form (17) and λ is the Bäcklund parameter. Using the Cole–Hopf transformation

$$\hat{v} = v + \frac{\psi_x}{\psi}$$

we obtain a third-order linear spectral problem [10]

$$\psi_{xxx} - \frac{\hat{v}_{xx}}{\hat{v}}\psi_x + \lambda\psi = 0. \tag{18}$$

Note that the Tzitzeica equation (17) is invariant under the permutation $P(\partial_x, \partial_t) \to (\partial_t, \partial_x)$. So, the 'time part' of the Lax pair is given by a third-order spectral problem where x is replaced by t and λ by $1/\lambda$.

3.3. The Fitzhugh–Nagumo equation

A similar result can be obtained for the Fitzhugh–Nagumo equation [12]

$$v_t - v_{xx} + v(1-v)(a-v) = 0 \qquad -1 \leqslant a \leqslant 1. \tag{19}$$

Writing the 2-Riccati equation (6) for the field v we can show that equation (6) is compatible with equation (19) iff:

$$c = 2^{-1/2} \qquad \alpha_2 = -2^{-1/2}(1+a) \qquad \alpha_1 = 2^{-1}a \tag{20}$$

$$\alpha_0 = \hat{v}_{xx} + 3 \times 2^{-1/2}\hat{v}\hat{v}_x + \tfrac{5}{2}\hat{v}^3 + 2^{-1/2}(1+a)\hat{v}_x + \tfrac{1}{2}\hat{v}^2(1+a) - \tfrac{1}{2}a\hat{v} \tag{21}$$

where $\hat{v}$ is another solution of equation (17). The 2-Riccati (20) is associated with a third-order spectral problem

$$\psi_{xxx} - 2^{-1/2}(1+a)\psi_{xx} + 2^{-1}a\psi_x + \left[-\hat{v}_{xx} + 3 \times 2^{-1/2}\hat{v}\hat{v}_x + \tfrac{5}{2}\hat{v}^3 + 2^{-1/2}(1+a)\hat{v}_x + \tfrac{1}{2}\hat{v}\right]\psi = 0 \tag{22}$$

and the corresponding Darboux transformation [12] is given by

$$v = \hat{v} + 2^{1/2}\frac{\psi_x}{\psi}. \tag{23}$$

For $\hat{v} = 0$ we have

$$\psi = A(t)\,\mathrm{e}^{-2^{-1/2}\cdot x} + B(t)\,\mathrm{e}^{-2^{-1/2}ax}$$

from which we obtain the 'soliton' solution [15]

$$v = v_0 - a^{1/2}\hat{v}x - \frac{22^{1/2}a}{a-1}(1-\hat{v}^2)^{1/2}\ln\left|\cosh\left(\frac{a-1}{22^{1/2}}x + \varphi(t)\right)\right|.$$

Let us notice, however, that both the Bäcklund transformation (20) and the spectral problem (22) are free of a spectral parameter, usually an indication of nonintegrability.

4. Conclusions

In this short paper we have shown that there is a strong relationship between Riccati equations and Bäcklund transformations for integrable nonlinear partial differential equations. As has been established in many of the well known cases [14], the simplest Bäcklund transformation is given by the classical first-order Riccati equation. There are, however, a few well known cases in which the simplest Bäcklund transformation is given by a higher-order differential equation. We have demonstrated by a few examples that in such a case the Bäcklund transformation is given by a higher Riccati equation, higher in the so-called Riccati chain. Not all known cases of fifth-order equations have Bäcklund transformations having the form of one of the Riccati chain equations. For example the Kaup–Kupershmidt equation [11] has a Bäcklund transformation [17] given by a second order differential equation, which is not equivalent to the second-order equation of the Riccati chain but it reduces to it by a contact transformation [5]. The Riccati chain is obtained from the $Sl(N, R)$ matrix Riccati equation [6] with the restriction that its linear problem can be written as a scalar differential equation. By different reduction of the matrix Riccati equation we can think of constructing different Riccati chains.

Bäcklund transformations can be thought of as conditional symmetries for the equation under study [18], i.e. symmetries of the overdetermined system obtained by adding to the given differential equation under investigation differential constraints for which the symmetry criterion is identically satisfied. Up to now the only set of conditions for which the symmetry criterion is identically satisfied is given by first-order differential equations [19]. The fact, shown here, that higher-order Riccati equations may also play a role in the construction of Bäcklund transformations for some nonlinear partial differential equations indicates the possibility of introducing higher-order conditional symmetries. This result can open the way to the construction of new classes of exact solutions for many physically important differential equations [20, 21]. Work on the extension of these results to the case of matrix Riccati chains and their reduction in application to nonlinear partial differential equations is in progress.

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